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MORE-THAN-NEARLY-PERFECT PACKINGS AND PARTIAL DESIGNS

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Results of Frankl and Rödl and of Pippenger and Spencer show that uniform hypergraphs which are almost regular and have small maximal pair degrees (codegrees) contain collections of pairwise disjoint edges (packings) which cover all but $o(n)$ of the n vertices. Here we show, in particular, that regular uniform hypergraphs for which the ratio of degree to maximum codegree is n^{ε}, for some $\varepsilon > 0$, have packings which cover all but $n^{1-\alpha}$ vertices, where $\alpha = \alpha(\varepsilon) > 0$.

The proof is based on the analysis of a generalized version of Rödl's nibble technique.

We apply the result to the problem of finding partial Steiner systems with almost enough blocks to be Steiner systems, where we prove that, for fixed positive integers $t < k$, there exist partial $\mathrm{S}(t,k,n)$'s with at most $n^{t-1/(2\binom{k}{t}-1)+o(1)}$ uncovered t-sets, improving the earlier $o(n^t)$ result.

1. Introduction

In [3] Frankl and Rödl showed that every hypergraph satisfying certain regularity conditions contains a collection of pairwise disjoint edges which covers, in a specific sense, almost every vertex of the hypergraph. Our purpose in this paper is to present a more exact analysis of the situation and to show that, under slightly stronger conditions, there exist packings with many fewer uncovered vertices. The additional conditions are not so strong as to exclude any of the important examples.

We start by reviewing the essential definitions and sketching the history of the problem. A *hypergraph* H is a set $V(H)$, whose elements are called *vertices*, and a set $E(H)$ of subsets of $V(H)$, whose elements are called *edges*. A hypergraph is *k-uniform* if each of its edges contains exactly k vertices. In this paper, we allow k to be a non-decreasing integer-valued function of the number of vertices.

The *degree* of a vertex v, denoted $\deg(v)$, is the number of edges containing that vertex and the *codegree* of two vertices u and v, denoted $\mathrm{codeg}(u,v)$, is the number of edges containing both of them. Often, we are not interested so much in the degree of a particular vertex, but in the degree of the vertex with the largest degree. Therefore, we define

$$\mathrm{maxdeg}(H) = \max_{v \in V(H)} \deg(v).$$

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We also define

$$\mathrm{mindeg}(H) = \min_{v \in V(H)} \deg(v),$$

$$\mathrm{avgdeg}(H) = \frac{1}{|V(H)|} \sum_{v \in V(H)} \deg(v) = \frac{k|E(H)|}{|V(H)|},$$

and

$$\mathrm{maxcodeg}(H) = \max_{u,v \in V(H), u \neq v} \mathrm{codeg}(u,v).$$

A hypergraph is *D-regular* if all of its vertices have degree exactly D.

A *packing* in a hypergraph is a collection of pairwise disjoint edges. When $k=2$, the 2-uniform hypergraphs are usually called *graphs* and the packings are usually called *matchings*. We say that a particular vertex is *covered* by a given packing if that vertex is contained in one of the edges of that packing. We are particularly interested in packings which cover almost all of the vertices. For a positive real ε, we call a packing *ε-nearly-perfect* if it leaves at most εn of the n vertices uncovered.

Often it will be convenient to work asymptotically. Naturally, we will use standard asymptotic notation: for two real-valued functions f and g of n, we write $f=o(g)$ or $f \ll g$ if $\lim_{n\to\infty} f/g=0$ and $f=O(g)$ if $\limsup_{n\to\infty} f/g$ is finite. With this notation, a *nearly-perfect* packing may be defined as a packing in which only $o(n)$ of the n vertices are left uncovered.

As a precursor to [3], Rödl proved in [8] that nearly-perfect partial $\mathrm{S}(t,k,n)$ Steiner systems exist for fixed t and k, settling an old conjecture of Erdős and Hanani [2]. (This result and a strengthening following from our result will be discussed in section 6.) Frankl and Rödl [3] then showed that, under some conditions on the maximum codegree, for k fixed, k-uniform nearly-regular hypergraphs always contain nearly-perfect packings.

Pippenger and Spencer [7] weakened the codegree condition to maxcodeg $\ll$ maxdeg and, most importantly, used this result to show that the edges can be partitioned into packings, almost all of which are nearly-perfect. This implies that the chromatic index of such a hypergraph is asymptotic to its maximum degree, the minimum possible value. Recently, Kahn [5] improved this further, showing that a hypergraph with edge sizes bounded by a constant and maxcodeg $\ll$ maxdeg has list chromatic index asymptotic to its maximum degree.

Pippenger and Spencer's strengthening of Frankl and Rödl's nearly-perfect packing result may be formulated as follows.

Theorem 1.1. ([7, 3]) *Let k be a fixed positive integer. If H is a k-uniform hypergraph on n vertices such that*

$$\mathrm{mindeg}(H) = (1 - o(1))\,\mathrm{maxdeg}(H)$$

and

$$\mathrm{maxcodeg}(H) \ll \mathrm{maxdeg}(H)$$

then H contains a packing with at most $o(n)$ uncovered vertices.

All asymptotic statements should be read as n goes to infinity, so in a sense the theorem isn't about a single hypergraph H, but rather about an infinite family of hypergraphs.

Our purpose here is to present an improved analysis which shows that if the hypergraph (family) is regular, there exists a packing with many fewer uncovered vertices. We also include the more general case where k is a function of n.

Theorem 1.2. *If H is a D-regular k-uniform hypergraph on n vertices with maximum codegree C such that*

$$kC \log n \ll D \tag{1.1}$$

then H contains a packing with at most

$$n\left[\frac{kC\log n}{D}\right]^{1/(2k-1+o(1))}$$

uncovered vertices.

The same conclusion can be reached even if the hypergraph is slightly non-regular.

It is worth noting that all of the examples given in [3] and [7] as applications of Theorem 1.1 also satisfy the conditions of Theorem 1.2. This results in improvements all around. It should also be possible to improve the results on the chromatic and list chromatic indices of a hypergraph [7, 5] and Kahn's asymptotic version of the Erdős–Faber–Lovasz Conjecture [4].

Note that based on the original version of this paper, written in 1994, Alon, Kim, and Spencer [1] were able to improve Theorem 1.2 in the case when k is fixed and $\mathrm{maxcodeg}(H)=1$ (H is *simple*). They can show that, in this case, when $k=3$, H will contain a packing with at most $O(nD^{-1/2}\log^{3/2} n)$ uncovered vertices and, when $k>3$, H will contain a packing with at most $O(nD^{-1/(k-1)})$ uncovered vertices.

This paper is organized as follows. In the next section an important large deviation inequality, which is probably of independent interest, will be developed. The following three sections will present the proof of Theorem 1.2, and the last section will apply the the theorem to the problem of finding partial t-designs with very few uncovered t-sets.

2. Large deviations

The proof of our main result relies heavily on probabilistic techniques. Generally, when using such techniques one often needs good large deviation results. In this section we apply a recent result of Talagrand [10] to prove a useful large deviation inequality. A related, but different, treatment of large deviation inequalities derived from Talagrand's work may be found in [9].

Throughout this section let n be a positive integer, $\Omega = \prod_{1\le i\le N}\Omega_i$ be a product probability space. For a non-empty subset A of Ω and a positive real t, define the set $A_t\subseteq\Omega$ as the set of all $y\in\Omega$ such that, for every unit vector $\alpha\in\mathbf{R}^N$, there exists an $x\in A$ such that

$$\sum_{i \text{ s.t. } x_i\neq y_i} \alpha_i < t.$$

Talagrand proves the following result.

Theorem 2.1. (Talagrand's Inequality.) *If* $A\subseteq\Omega$ *and* $t>0$ *then* $\Pr[A]\cdot\Pr[\overline{A_t}]\le e^{-t^2/4}$.

Please note that we will be using multisets and will count cardinalities with mulitplicity. Our first theorem is similar to Talagrand's Theorem 7.1.2.

Theorem 2.2. *Let* $L:\Omega\to\mathbf{R}$ *be such that for every* $x\in\Omega$ *there exists a multiset* J_x *(a certificate) whose elements come from* $\{1,\ldots,N\}$ *with maximum multiplicity* b *such that* $|J_x|=L(x)$ *and such that for each* $y\in\Omega$, $L(y)\ge|\{i\in J_x\,|\,x_i=y_i\}|$. *For a median* M *of* L *and* $u>0$,

$$\Pr[L(x)\ge M+u]\le 2\exp\left\{-\frac{u^2}{4b(M+u)}\right\}$$

and

$$\Pr[L(x)\le M-u]\le 2\exp\left\{-\frac{u^2}{4bM}\right\}.$$

Proof. Let $A=\{z\in\Omega\,|\,L(z)\le M\}$, so that $\Pr[A]\ge 1/2$, and let $t=u\cdot[b(M+u)]^{-1/2}$. Select an x in Ω such that $L(x)\ge M+u$. We'll prove that $\{x\,|\,L(x)\ge M+u\}\subseteq\overline{A_t}$, which will imply by Talagrand's inequality that

$$\begin{aligned}\Pr[L(x)\ge M+u]&\le\Pr[\overline{A_t}]\le 2\Pr[A]\cdot\Pr[\overline{A_t}]\\&\le 2e^{-t^2/4}=2\exp\left\{-\frac{u^2}{4b(M+u)}\right\},\end{aligned}$$

as desired.

Let J_x be a certificate for x. Now define a unit vector α in the following way: let $\hat\alpha_i$ be the multiplicity of i in J_x and normalize $\hat\alpha$ to get α. Since $|J_x|=L(x)$ and J_x has multiplicity bounded by b, the normalization factor $\sqrt{|J_x|}$ is at most $\sqrt{bL(x)}$.

Suppose x were in A_t. Then for this α in particular, there exists a $y \in A$ (a $y \in \Omega$ with $L(y) \leq M$) with $\sum_{x_i \neq y_i} \alpha_i < t$. Denormalizing, this implies that $|\{i \in J_x \mid x_i \neq y_i\}| < t\sqrt{bL(x)}$.

Since J_x is a certificate for x,

$$\begin{aligned} M \geq L(y) &\geq |\{i \in J_x \mid x_i = y_i\}| = L(x) - |\{i \in J_x \mid x_i \neq y_i\}| \\ &> L(x) - t\sqrt{bL(x)} \geq (M+u) - t\sqrt{b(M+u)} = M, \end{aligned}$$

a clear contradiction. This implies that $x \notin A_t$ and therefore that $\{x \mid L(x) \geq M + u\} \subseteq \overline{A_t}$, so the bound on the upper tail is proven.

To prove the bound on the lower tail, set $A = \{z \in \Omega \mid L(z) \leq M - u\}$ and $t = u \cdot [bM]^{-1/2}$. Prove, again by contradiction, that $L(x) \geq M$ implies that $x \notin A_t$. This implies that $\Pr[\overline{A_t}] \geq \Pr[L(x) \geq M] \geq 1/2$, so

$$\begin{aligned} \Pr[L(x) \leq M - u] = \Pr[A] &\leq 2\Pr[A] \cdot \Pr[\overline{A_t}] \\ &\leq 2e^{-t^2/4} = 2\exp\left\{-\frac{u^2}{4bM}\right\}, \end{aligned}$$

as desired. ∎

The main difficulty with Theorem 2.2 is that it is a large deviation result about a median of L rather than about the mean. The next proposition gives us some idea about how far apart the two may lie.

Proposition 2.3. *If L is a function satisfying the conditions of Theorem 2.2 and M is a median of L then*

$$M - 2\sqrt{\pi bM} \leq \mathrm{Ex}[L(x)] \leq M + 2\sqrt{2\pi bM} + 8b.$$

Proof. To prove the lower bound, observe that for any $u > 0$,

$$\begin{aligned} &\mathrm{Ex}[L(x)] \geq (M-u)\Pr[L(x) \geq M-u] \\ &\qquad + \sum_{i\geq 2}(M - iu)\Pr[M - iu \leq L(x) < M - (i-1)u] \\ &= (M-u)\big(1 - \Pr[L(x) < M-u]\big) \\ &\qquad + \sum_{i\geq 2}(M-iu)\big(\Pr[L(x) < M-(i-1)u] - \Pr[L(x) < M - iu]\big) \\ &= (M-u) - u\sum_{i\geq 1}\Pr[L(x) < M - iu] \\ &\geq M - u - u\sum_{i\geq 1} 2\exp\left\{-\frac{(iu)^2}{4bM}\right\} \end{aligned}$$

by Theorem 2.2. Taking the limit as u goes to 0 from above gives

$$\mathrm{Ex}[L(x)] \geq M - 2\int_0^\infty \exp\left\{-\frac{x^2}{4bM}\right\} dx = M - 2\sqrt{\pi b M}.$$

On the other hand, for any $u > 0$,

$$\begin{aligned}
\mathrm{Ex}[L(x)] &\leq (M+u)\Pr[L(x) \leq M+u] \\
&\quad + \sum_{i\geq 2}(M+iu)\Pr[M+(i-1)u < L(x) \leq M+iu] \\
&= (M+u)\big(1-\Pr[L(x) > M+u]\big) \\
&\quad + \sum_{i\geq 2}(M+iu)\big(\Pr[L(x) > M+(i-1)u] - \Pr[L(x) > M+iu]\big) \\
&= (M+u) + u\sum_{i\geq 1}\Pr[L(x) > M+iu] \\
&\leq M+u+u\sum_{i\geq 1} 2\exp\left\{-\frac{(iu)^2}{4b(M+iu)}\right\}.
\end{aligned}$$

Again taking the limit as u goes to 0 from above gives

$$\begin{aligned}
\mathrm{Ex}[L(x)] &\leq M + 2\int_0^\infty \exp\left\{-\frac{x^2}{4b(M+x)}\right\} dx \\
&\leq M + 2\int_0^M \exp\left\{-\frac{x^2}{8bM}\right\} dx + \int_M^\infty \exp\left\{-\frac{x-M}{4b}\right\} dx \\
&\leq M + 2\sqrt{2\pi b M} + 8b
\end{aligned}$$

as desired. ∎

Equivalently, this proposition states that

$$\mu + (4\pi - 8)b - 2\sqrt{2\pi b(\mu + (2\pi - 8)b)} \leq M \leq \mu + 2\pi b + 2\sqrt{\pi b(\mu + \pi b)}$$

where $\mu = \mathrm{Ex}[L(x)]$.

Translated into asymptotic language, the proposition says that if $b \ll \mu$ then

$$M = \mu + O\left(\sqrt{b\mu}\right)$$

and Theorem 2.2 says that if $\sqrt{bM} \ll u \ll M$ then

$$\Pr[|L(x) - M| \geq u] \leq \exp\left\{-(1+o(1))\frac{u^2}{4bM}\right\}.$$

Combining these two statements, we get the following corollary, which we will employ frequently.

Corollary 2.4. *Let L be as in Theorem 2.2 and let $\mu = \text{Ex}[L(x)]$. If $1 \ll \varphi \ll \mu/b$ then*

$$\Pr\left[|L(x) - \mu| \geq (1 + o(1))2\sqrt{b\mu\varphi}\right] \leq e^{-\varphi}.$$

Notice that we swapped the emphasis: rather than asking what error probability we get for a given concentration, we ask what concentration we can guarantee with a given error probability. This reflects the fact that when we use the corollary, we will want the tightest concentration possible.

3. Packings and the nibble algorithm

The basic technique used to prove our main result goes back to Rödl's original paper [8]. The idea is to use an iterative, randomized algorithm to slowly construct a packing while leaving the unpacked portion of the hypergraph fairly homogeneous. The version we use is a generalization of the one given in [7].

Rödl Nibble Algorithm

Inputs: An integer k, a k-uniform hypergraph H, a positive real number η, and a positive integer t.

Output: A packing in H.

$H_0 \leftarrow H$.
$D_0 \leftarrow \text{avgdeg}(H)$.
for $i \leftarrow 1$ to t do
  $X_i \leftarrow$ edges selected independently at random from $E(H_{i-1})$ with probability η/D_{i-1}.
  $Y_i \leftarrow$ edges in X_i which are disjoint from all other edges in X_i.
  $V(H_i) \leftarrow$ vertices from $V(H_{i-1})$ which are not contained in any edge in X_i.
  $E(H_i) \leftarrow$ edges from $E(H_{i-1})$ which are completely contained within $V(H_i)$.
  $D_i \leftarrow e^{-\eta(k-1)}D_{i-1}$.
output $Y_1 \cup Y_2 \cup \ldots \cup Y_t$.

We call each iteration of the loop a *nibble*. The idea is that during each nibble a few, mostly disjoint edges and everything which touches them are removed from the hypergraph. In the end, those selected edges which were disjoint from the others form the desired packing.

Clearly this algorithm generates a packing regardless of the structure of H and the choices of η and t. We aim to show that if H is regular and η and t are chosen correctly then the resulting packing will cover most of the vertices.

This is, in fact, what Pippenger and Spencer did in [7] for the case where k is fixed and H satisfies the conditions $\text{mindeg}(H) = (1 - o(1))\,\text{maxdeg}(H)$ and $\text{maxcodeg}(H) \ll \text{maxdeg}(H)$. They show that if $\alpha < 1$ is *fixed* and η and t are chosen to satisfy

$$\alpha = \eta e^{-\eta k} \frac{1 - e^{-\eta t}}{1 - e^{-\eta}}$$

(so η and t are also fixed) then, with probability $1-o(1)$, the algorithm produces a packing which covers $\alpha|V(H)|$ vertices of H. Therefore asymptotically there exist packings which cover $(1-o(1))|V(H)|$ vertices.

We can do even better than that. We will eventually find a method for choosing η and t and conditions on k and H which ensure that the resulting packing has many fewer uncovered vertices. The exact number will depend on k and to some extent on H—namely, the ratio between its maximum codegree and its average degree. What is more, our result is not restricted to (classes of) hypergraphs with constant edge size. Under the right conditions, we can show the existence of near-perfect (or better) packings even when k grows with the number of vertices.

It should be noted that if one used only the techniques of [7], one could not improve the results no matter what additional restrictions were placed on H. Thus new techniques and tools such as the large deviation result of the previous section are required.

In the next section we analyse what happens during a single nibble. Then, in the following section, we analyse the behaviour of a sequence of nibbles, producing our main result.

4. One nibble

What happens when a hypergraph is subjected to a single nibble? To simplify notation, let H be the initial hypergraph and let H' be the result of the nibble. Also let D be a positive real. H, H', and D correspond to H_{i-1}, H_i, and D_{i-1}, respectfully, in the previous and subsequent sections. We also denote by C an upper bound of the maximum codegree of H. In this section, t will play no role, but η will be an important (and as yet unspecified) parameter.

Our main goal in this section is to show that if the vertex degrees of H are all close to D then the vertex degrees of H' are all close to $e^{-\eta(k-1)}D$ (which corresponds to D_i). It might be that some vertex's degree deviates from this ideal value, but we'll show that this happens only extremely rarely.

A second goal will be to measure two sources of "error" in the packing created by the algorithm. Specifically, vertices which are uncovered at the end come from two sources: vertices contained in edges which were selected but which intersect another selected edge (which we call *clashed edges*) and vertices which survive all the way through t nibbles. In this section we'll determine for a single nibble how many vertices are contained in clashed edges and how many survive.

In order to quantify the irregularity of H, let ε be such that

$$(1 - \varepsilon)D \le \operatorname{mindeg}(H) \le \operatorname{maxdeg}(H) \le (1 + \varepsilon)D.$$

It might be helpful to think of the quantities ε, k, C, and D as functions of H. We will put additional constraints on these quantities which will be reflected later as constraints on H and will eventually lead to specifications for η and t.

At this point, we make two assumptions:

$$\frac{kC}{D} \ll \varepsilon \ll 1 \qquad \text{and} \qquad \eta k \ll 1. \tag{4.1}$$

Together these imply that $\eta/D \ll \varepsilon$ and that $1 \ll \varepsilon D$.

The first result is a lemma which deals with the probability that a collection of vertices survive, given that some others are known to have survived.

Lemma 4.1. *Let ε, η, k, C, D, and H be as in this section. If $j = O(k)$ and $\ell = O(k)$ are positive integers and $v_1, v_2, \ldots v_j, u_1, u_2, \ldots, u_\ell$ are distinct vertices of H then*

$$\Pr[v_1, \ldots, v_j \in V(H') \mid u_1, \ldots, u_\ell \in V(H')] = e^{-\eta j}\Big(1 \pm (1 + o(1))\varepsilon\eta j\Big),$$

where $(1 \pm x)$ denotes some value between $(1 - x)$ and $(1 + x)$.

Proof. In this proof we require some routine inequalities: for $|a| \le b$ and $b \ge 1$,

$$e^{-a}\Big(1 - \frac{a^2}{b}\Big) \le \Big(1 - \frac{a}{b}\Big)^b \le e^{-a};$$

for $0 \le a \le 1$ and $b \ge 1$,

$$(1 - a)^b \ge 1 - ab;$$

and, for positive a and b with $ab \le 1$,

$$(1 - a)^{-b} \le 1 + ab + (ab)^2.$$

Let T be the number of edges of H adjacent to at least one of the v_i's but not adjacent to any of the u_i's. Since the degree of each vertex is between $(1 - \varepsilon)D$ and $(1 + \varepsilon)D$ and the codegree of each pair of vertices is at most C,

$$(1 - \varepsilon)jD - \left(\binom{j}{2} + j\ell\right) C \le T \le (1 + \varepsilon)jD.$$

With assumption (4.1), this is

$$\big[1 - (1 + o(1))\varepsilon\big]jD \le T \le \big[1 + \varepsilon\big]jD.$$

Since all edges adjacent to at least one of the u_i's must not have been selected, the quantity we're interested in

$$\Pr[v_1, \ldots, v_j \in V(H') \mid u_1, \ldots, u_\ell \in V(H')] = \Big(1 - \frac{\eta}{D}\Big)^T.$$

We look first at the upper bound. By the lower bound on T,

$$\begin{aligned}\left(1-\frac{\eta}{D}\right)^T &\le \left(1-\frac{\eta}{D}\right)^{jD-(1+o(1))\varepsilon jD} \le e^{-\eta j}\left(1-\frac{\eta}{D}\right)^{-(1+o(1))\varepsilon jD}\\ &\le e^{-\eta j}\Big(1+(1+o(1))\varepsilon\eta j\Big).\end{aligned}$$

The lower bound is equally easy. By the upper bound on T,

$$\begin{aligned}\left(1-\frac{\eta}{D}\right)^T &\ge \left(1-\frac{\eta}{D}\right)^{jD+\varepsilon jD} \ge e^{-\eta j}\left(1-\frac{\eta^2}{D}\right)^j\left(1-\frac{\eta}{D}\right)^{\varepsilon jD}\\ &\ge e^{-\eta j}\left(1-\frac{\eta^2 j}{D}\right)(1-\varepsilon\eta j) = e^{-\eta j}\Big(1-(1+o(1))\varepsilon\eta j\Big).\end{aligned}$$ ∎

Our first major proposition shows that the vertex degrees after the nibble remain highly concentrated—that is, the graph remains almost as regular as it was. In order to express (and control) the probability of a deviation in the degrees, we introduce the parameter φ, a positive, real-valued function of H. Later, when we use this proposition, φ will be on the order of the logarithm of the number of vertices of H.

Proposition 4.2. *Let ε, η, k, C, D, and H be as in this section. If v is a fixed vertex of H' and φ is a positive, real-valued function of H such that $1 \ll \varphi \ll \varepsilon^2\eta D/C$ then, with probability at least $1-e^{-\varphi}$,*

$$\deg_{H'}(v) = e^{-\eta(k-1)}D\Big(1\pm\varepsilon\big[1+(1+o(1))\eta(k-1)\big]\Big).$$

Proof. Begin by considering a fixed edge e incident with v. Applying Lemma 4.1 with $\ell=1$, $u_1=v$, $j=k-1$ and the v_i's being the remaining vertices of e, we see that

$$\Pr[e\in E(H') \mid v\in V(H')] = e^{-\eta(k-1)}\Big(1\pm(1+o(1))\varepsilon\eta(k-1)\Big).$$

Therefore, summing over all edges e,

$$\mathrm{Ex}[\deg_{H'}(v)] = e^{-\eta(k-1)}\deg_H(v)\Big(1\pm(1+o(1))\varepsilon\eta(k-1)\Big) \tag{4.2}$$

and, since $\deg_H(v)=(1\pm\varepsilon)D$,

$$\begin{aligned}\mathrm{Ex}[\deg_{H'}(v)] &= e^{-\eta(k-1)}D(1\pm\varepsilon)\Big(1\pm(1+o(1))\varepsilon\eta(k-1)\Big)\\ &= e^{-\eta(k-1)}D\Big(1\pm\varepsilon\big[1+(1+o(1))\eta(k-1)\big]\Big).\end{aligned} \tag{4.3}$$

To show that $\deg_{H'}(v)$ is concentrated about its expectation, we now apply Corollary 2.4. Number the edges intersecting at least one edge incident with v but not containing v itself from 1 to n and let Ω be the probability space formed by the selection choices made on those edges. For a given $x\in\Omega$, that is, a given selection of edges, let $L(x)$ be the number of edges incident with v which intersect at least one selected edge. In other words, let $L=\deg_H(v)-\deg_{H'}(v)$.

For a given $x\in\Omega$, we can form a multiset (certificate) J_x such that $|J_x|=L(x)$ and such that for each $y\in\Omega$, $L(y)\geq|\{i\in J_x\,|\,x_i=y_i\}|$ as required by the corollary by selecting, for each edge incident with v which intersects at least one selected edge, one of the intersecting selected edges. The maximum multiplicity of such a set is the maximum number of edges incident with v and intersecting a given edge—that is, $b=kC$.

Corollary 2.4 states that if $1\ll\varphi\ll\mu/b$ then

$$\Pr\left[|L(x)-\mu|\geq(1+o(1))2\sqrt{b\mu\varphi}\right]\leq e^{-\varphi},$$

where $\mu=\mathrm{Ex}[L(x)]=\deg_H(v)-\mathrm{Ex}[\deg_{H'}(v)]$. Since $|L(x)-\mu|=|\deg_{H'}(v)-\mathrm{Ex}[\deg_{H'}(v)]|$,

$$\Pr\left[\left|\deg_{H'}(v)-\mathrm{Ex}[\deg_{H'}(v)]\right|\geq(1+o(1))2\sqrt{kC\mu\varphi}\right]\leq e^{-\varphi}. \tag{4.4}$$

And how large is μ? Looking back to (4.2) and using assumption (4.1), we see that

$$\begin{aligned}\mu&=\deg_H(v)-\mathrm{Ex}[\deg_{H'}(v)]\\&=\deg_H(v)\left[1-e^{-\eta(k-1)}\Big(1\pm(1+o(1))\varepsilon\eta(k-1)\Big)\right]\\&=(1+o(1))D\Big[1-\big(1-\eta(k-1)+o(\eta(k-1))\big)\big(1\pm o(\eta(k-1))\big)\Big]\\&=(1+o(1))\eta(k-1)D.\end{aligned}$$

Therefore, (4.4) says that $\deg_{H'}(v)$ is within

$$(1+o(1))2\sqrt{\eta k(k-1)CD\varphi}$$

of its expectation with probability at least $1-e^{-\varphi}$, for $1\ll\varphi\ll\eta D/C$. Combining this with (4.3) gives that

$$\deg_{H'}(v)=e^{-\eta(k-1)}D\Big(1\pm\varepsilon\big[1+(1+o(1))\eta(k-1)\big]\Big)\pm(1+o(1))2\sqrt{\eta k(k-1)CD\varphi}$$

with probability at least $1-e^{-\varphi}$.

Further, if we tighten the upper bound on φ to $\varphi\ll\varepsilon^2\eta D/C$, we see that

$$(1+o(1))2\sqrt{\eta k(k-1)CD\varphi}\ll\varepsilon\eta kD\leq O\left(e^{-\eta(k-1)}D\cdot\varepsilon\eta(k-1)\right),$$

so

$$\deg_{H'}(v)=e^{-\eta(k-1)}D\Big(1\pm\varepsilon\big[1+(1+o(1))\eta(k-1)\big]\Big)$$

with probability at least $1-e^{-\varphi}$, for $1 \ll \varphi \ll \varepsilon^2\eta D/C$, as claimed in the proposition. ∎

Proposition 4.3. *Let ε, η, k, C, D, and H be as in this section. The expected number of vertices in H' is at most $e^{-\eta}\big(1+(1+o(1))\varepsilon\eta\big)|V(H)|$.*

Proof. For a specific vertex $v \in V(H)$, Lemma 4.1 says that

$$\Pr[v \in V(H')] = e^{-\eta}\Big(1 \pm (1+o(1))\varepsilon\eta\Big),$$

from which the proposition follows by summing over all vertices of H. ∎

Proposition 4.4. *Let ε, η, k, C, D, and H be as in this section. The expected number of vertices contained in clashed edges is at most $(1+\varepsilon)^2\eta^2 k|V(H)|$.*

Proof. For an edge e of H,

$$\Pr[e \text{ is clashed}] = \frac{\eta}{D}\left[1-\left(1-\frac{\eta}{D}\right)^T\right],$$

where T is the number of edges intersecting e. Since $\operatorname{maxdeg}(H) \le (1+\varepsilon)D$, T is at most $(1+\varepsilon)kD$ and hence

$$\left(1-\frac{\eta}{D}\right)^T \ge \left(1-\frac{\eta}{D}\right)^{(1+\varepsilon)kD} \ge 1-(1+\varepsilon)\eta k$$

which means that

$$\Pr[e \text{ is clashed}] \le (1+\varepsilon)\frac{\eta^2 k}{D}.$$

Since each vertex has at most $(1+\varepsilon)D$ incident edges, the probability that a given vertex is contained in a clashed edge is at most $(1+\varepsilon)^2\eta^2 k$. Therefore the expected number of vertices covered by clashed edges is at most $(1+\varepsilon)^2\eta^2 k|V(H)|$. ∎

5. Many nibbles

Now we return to the analysis of the entire Rödl Nibble algorithm. Recall that the algorithm has two parameters η and t and generates a sequence $H = H_0 \supseteq H_1 \supseteq \ldots \supseteq H_t$ of k-uniform hypergraphs and a packing in H. Our goal here is to find conditions on k and H and specifications for η and t such that the algorithm almost always results in a packing with very few uncovered vertices.

Uncovered vertices must be of one of only two types: vertices contained in H_t (the last hypergraph) and vertices contained in clashed edges (intersecting edges both selected during the same nibble). It is therefore essential that we have tight bounds on these numbers.

Degree regularity will play an important role. As long as the hypergraph remains fairly regular, we can use Propositions 4.3 and 4.4 to find decent bounds on the number of vertices in H_t and the number of vertices contained in clashed edges. Specifically, we would like to find choices for ε_i so that if $H = H_0$ satisfies

$$(1-\varepsilon_0)D_0 \le \mathrm{mindeg}(H_0) \le \mathrm{maxdeg}(H_0) \le (1+\varepsilon_0)D_0$$

then, for each $1 \le i \le t$, H_i satisfies

$$(1-\varepsilon_i)D_i \le \mathrm{mindeg}(H_i) \le \mathrm{maxdeg}(H_i) \le (1+\varepsilon_i)D_i$$

(recall that the algorithm sets $D = D_0$ to the average degree of H and $D_i = e^{-\eta(k-1)}D_{i-1} = e^{-\eta(k-1)i}D$).

In order to use the results in the previous section we also need assumption (4.1). Since the ε_i grow with i, the assumption $\varepsilon_t \ll 1$ places an upper bound on t in terms of H and η. This in turn means that H_t will have more vertices. It is therefore clear that we want the smallest possible values for the ε_i, subject to our assumptions.

In order to simplify things, we won't keep track of C_i, but merely note that the C_i's must decrease as i increases and always use the bound $C_i \le C_0 = C$.

Proposition 4.2 tells us that if v is a vertex of H_{i-1} and φ is such that $1 \ll \varphi \ll \varepsilon_{i-1}^2 \eta D_{i-1}/C$ then with probability at least $1 - e^{-\varphi}$,

$$\deg_{H_i}(v) = D_i\Big(1 \pm \varepsilon_{i-1}\big[1 + (1+o(1))\eta(k-1)\big]\Big),$$

which means that the best we might do is put

$$\varepsilon_i = \varepsilon_{i-1}\big[1 + (1+o(1))\eta(k-1)\big].$$

This recurrence is easily solved, giving

$$\varepsilon_i = \varepsilon_0 \cdot \big[1 + (1+o(1))\eta(k-1)\big]^i = \varepsilon_0\, e^{(1+o(1))\eta(k-1)i}.$$

For concreteness, define r to be the $(1+o(1))$ function appearing in the last expression. So the formal definition of ε_i is $\varepsilon_0\, e^{r\eta(k-1)i}$, where ε_0 will be specified later.

The next proposition summarizes our discussion to this point, giving simple conditions which imply that this choice of ε_i satisfies all of the necessary assumptions in order to use the propositions of the previous section, and then presents their conclusions.

Proposition 5.1. *Let $n = |V(H)|$ and $c > 0$ be an absolute constant. If these conditions hold:*

$$\log n \ll \frac{\varepsilon_0^2 \eta D}{C}, \quad \eta k \ll 1, \quad \varepsilon_0 \ll e^{-r\eta(k-1)t}, \quad \textit{and} \quad t \ll n^2 \tag{5.1}$$

then, with probability at least $1-n^{-c}$,
(a) for every $0\le i\le t$ *and every vertex* v *of* H_i, $\deg_{H_i}(v)=(1\pm\varepsilon_i)D_i$*;*
(b) the expected number of vertices in H_t *is at most* $(1+o(1))\,e^{-\eta t}n$*; and*
(c) the expected number of vertices contained in clashed edges during all t *steps is at most* $(1+o(1))\,\eta k n$*.*

Proof. First we show that condition (5.1) implies all previously made assumptions. Then we use the propositions from the previous section to prove the conclusions.

In order to use Proposition 4.2, we assume that there is some function φ such that

$$1\ll\varphi\ll\frac{\varepsilon_i^2\eta D_i}{C}=\frac{\varepsilon_0^2\eta D}{C}e^{(2r-1)\eta(k-1)i}. \tag{5.2}$$

The $i=0$ case gives the tightest constraint. As we shall see below, we can pick $\varphi=(c+3)\log n$, so the first part of (5.1) implies (5.2) for all i.

The second part of assumption (4.1), that $\eta k\ll 1$, is identical to the second part of (5.1).

The first part of assumption (4.1) is that, for each i,

$$\frac{kC}{D_i}\ll\varepsilon_i\ll 1.$$

The upper bound follows from the third part of (5.1) and the fact that the ε_i increase with i. Namely,

$$\varepsilon_i\le\varepsilon_t=\varepsilon_0\cdot e^{r\eta(k-1)t}\ll 1.$$

The lower bound follows from the fact that

$$\frac{\varepsilon_i D_i}{kC}=\frac{\varepsilon_i^2\eta D_i}{C}\cdot\frac{k}{\varepsilon_i\eta}\gg 1$$

since (5.2) says that $\frac{\varepsilon_i^2\eta D_i}{C}\gg 1$, $k\ge 1$, and $\varepsilon_i\eta\ll\eta\ll 1$. Thus all of our assumptions follow from (5.1) and we may proceed to the conclusions.

To prove conclusion (a), we apply Proposition 4.2 for each i and each v. Thus, in all, we need to apply the proposition at most $nt\ll n^3$ times, by the fourth part of (5.1). Since we want the conclusion to be true with probability at least $1-n^{-c}$, we need the probability in each invocation to be at least $1-n^{-c-3}$. Thus we need to set $\varphi=(c+3)\log n$, as we promised earlier.

Now that we know that in each H_i all vertex degrees are $(1\pm\varepsilon_i)D_i$, we can apply Propositions 4.3 and 4.4. Conclusion (b) follows from the repeated application of Proposition 4.3. More generally, for $0\le i\le t$,

$$\mathrm{Ex}[|V(H_i)|]=e^{-\eta i}n\prod_{j=0}^{i-1}\bigl(1+(1+o(1))\varepsilon_j\eta\bigr)$$

$$= e^{-\eta i} n \Big[1 + (1+o(1))\eta \sum_{j=0}^{i-1} \varepsilon_j\Big].$$

Since

$$\sum_{j=0}^{i-1} \varepsilon_j = \varepsilon_0 \sum_{j=0}^{i-1} e^{r\eta(k-1)j} = \varepsilon_0 \frac{e^{r\eta(k-1)i} - 1}{e^{r\eta(k-1)} - 1} \ll \frac{1}{\eta(k-1)},$$

by the second and third parts of (5.1),

$$\text{(5.3)} \qquad \mathrm{Ex}[|V(H_i)|] = e^{-\eta i} n \Big[1 + o\Big(\frac{1}{k-1}\Big)\Big] = (1+o(1))\, e^{-\eta i} n.$$

Conclusion (c) follows from Proposition 4.4 and (5.3): The expected number of vertices in clashed edges in all t steps is at most

$$\sum_{i=0}^{t-1} (1+\varepsilon_i)^2 \eta^2 k \,\mathrm{Ex}[|V(H_i)|] = \eta^2 k \sum_{i=0}^{t-1} (1+o(1))\, e^{-\eta i} n$$

$$= (1+o(1))\, \eta^2 kn \sum_{i=0}^{t-1} e^{-\eta i} \le (1+o(1))\, \eta^2 kn \frac{1}{1-e^{-\eta}} = (1+o(1))\, \eta kn,$$

as desired. ∎

Now we can prove Theorem 1.2. Use Proposition 5.1 with $c = 1$. This means that with probability $1 - 1/n$, the degrees of all vertices behave as desired and so the expected number of vertices left uncovered after t steps is

$$(1+o(1)) \left(e^{-\eta t} n + \eta kn\right).$$

Markov's inequality says that with probability at least $1/2$, the actual number of uncovered vertices is no more than twice the expectation, and so forth. In particular, with positive probability, the number of uncovered vertices is less than $2\left(e^{-\eta t} n + \eta kn\right)$.

We'd like to minimize this. We have not yet specified η and t, so we will do so in such a way as to minimize this number and simultaneously satisfy the conditions of (5.1).

It is reasonable at this point to "balance" the contributions of vertices in clashed edges and left-over vertices to the uncovered vertices. Setting $e^{-\eta t} n = \eta kn$ suggests that we should set

$$t = \frac{1}{\eta} \log \frac{1}{\eta k}.$$

Thus the first and third parts of (5.1) are equivalent to

$$\frac{C \log n}{\eta D} \ll \varepsilon_0^2 \ll (\eta k)^{2r(k-1)}.$$

Since our initial graph H is regular, it doesn't matter how we set ε_0. Looked at another way, if we set ε_0 to be something convenient, we're just requiring that the input graph be that regular. In any case, for there to be a suitable ε_0, this last inequality implies that η must satisfy

$$\eta \gg \frac{1}{k}\left[\frac{kC\log n}{D}\right]^{1/(2r(k-1)+1)}. \tag{5.4}$$

So we should define η to be a function of n which grows only slightly faster than the right-hand side.

How much faster? The basic condition (1.1) of Theorem 1.2 implies that $\frac{kC\log n}{D} \ll 1$ and by definition $r = 1+o(1)$, so (5.4) will be satisfied if we set

$$\eta = \frac{1}{k}\left[\frac{kC\log n}{D}\right]^{1/(2k-1+o(1))}$$

for a suitable function $o(1)$. The second part of (5.1), that $\eta k \ll 1$, follows directly.

The fourth part of (5.1), that $t \ll n^2$, is also easy. First, by definition,

$$\eta \geq \frac{1}{kD^{1/(2k-1+o(1))}} \geq \frac{1}{n^{1+1/(2k-1+o(1))}}.$$

Similarly,

$$\eta k \geq \frac{1}{D^{1/(2k-1+o(1))}} \geq \frac{1}{n^{1/(2k-1+o(1))}}.$$

Thus,

$$t = \frac{1}{\eta}\log\frac{1}{\eta k} \leq n^{1+1/(2k-1+o(1))}\log n^{1/(2k-1+o(1))} \ll n^2.$$

With this choice of η, we can now conclude that the number of uncovered vertices is no more than

$$4\eta k n = n\left[\frac{kC\log n}{D}\right]^{1/(2k-1+o(1))},$$

as claimed in the theorem.

6. Partial designs

In this section, we take a look at one application of our main result. It was, in fact, this problem which prompted Rödl to develop the first nibble algorithm [8]. Happily, we will be able to show somewhat more than he was able to at that time. We start with definitions and history.

A *t-design with blocks of size k* is a k-uniform hypergraph with the property that every set of t vertices is contained in exactly one of the edges (blocks) and a *partial* t-design with blocks of size k is a k-uniform hypergraph with the property that every set of t vertices is contained in *at most* one of the blocks. (Partial) t-designs with blocks of size k and n vertices (or *points*) are often called (partial) $\mathrm{S}(t,k,n)$ Steiner systems.

As was the case for packings, we are quite interested in partial designs which have almost enough blocks to be designs or, looked at another way, have blocks which cover almost every t-set. Call a partial $\mathrm{S}(t,k,n)$ *nearly-perfect* if at most $o\big(\binom{N}{T}\big)$ of the t-sets are uncovered (not contained in any block).

In [8], Rödl proved that when t and k are fixed, nearly-perfect partial $\mathrm{S}(t,k,n)$s exist, settling the Erdős-Hanani Conjecture [2]. We will use Thereom 1.2 to show that, when t and k are fixed, there exist partial $\mathrm{S}(t,k,n)$'s with many fewer uncovered t-sets.

The connection with packings in hypergraphs is that each (nearly-perfect) partial $\mathrm{S}(t,k,n)$ corresponds to a (nearly-perfect) packing in the following hypergraph, which we denote by $H(t,k,n)$. Let the vertices be the t-subsets of $\{1,\ldots,n\}$ and the edges correspond to the k-subsets in this manner: for each k-subset A, let its edge contain all those vertices whose t-subsets are subsets of A. In these terms, [8] proves that for fixed t and k, near-perfect packings exist in this class of hypergraphs.

Simple counting reveals that $H(t,k,n)$ has $\binom{n}{t}$ vertices, $\binom{n}{k}$ edges, and is $\binom{k}{t}$-uniform. Furthermore, it is regular of degree $\binom{n-t}{k-t}$ and has maximum codegree $\binom{n-t-1}{k-t-1}$.

We'll stick to the case where t and k are fixed positive integers, $t<k$. Theorem 1.2 does give a non-trivial result when t and k are not fixed, but the result is not as strong as the best known result [6] in this direction.

When we apply Theorem 1.2 to $H(t,k,n)$ (carefully, given that we've reused the symbols k and n), we see that condition (1.1) becomes

$$\binom{k}{t}\binom{n-t-1}{k-t-1}\log\binom{n}{t} \ll \binom{n-t}{k-t}.$$

This is equivalent to $\log n \ll n$, which is trivially true. The theorem then gives us the following corollary.

Corollary 6.1. *For fixed positive integers $t<k$, there exist partial $S(t,k,n)$s with at most*

$$\binom{n}{t}\left[O\left(\frac{\log n}{n}\right)\right]^{1/(2\binom{k}{t}-1+o(1))} = n^{t-1/(2\binom{k}{t}-1)+o(1)}$$

uncovered t-sets, improving the earlier $o(n^t)$ result.

This corollary doesn't say anything new in the case $t=2$ since Wilson proved in [11] that full $S(2,k,n)$s exist for fixed k and sufficiently large n, but for $t>2$ and large values of k it represents an improvement.

Alon, Kim, and Spencer's new result [1] gives a further improvement in the case $k=t+1$: there exist partial $S(t,t+1,n)$'s with at most $O(n^{t-1/t+o(1)})$ uncovered t-sets.

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